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Periodic heat conduction in a solid homogeneous finite cylinder

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Abstract

Analytic solution of the steady periodic, non-necessarily harmonic, heat conduction in a homogeneous cylinder of finite length and radius is given in term of Fourier transform of the fluctuating temperature field. The solutions are found for quite general boundary conditions (first, second and third kind on each surface) with the sole restriction of uniformity on the lateral surface and radial symmetry on the bases. The thermal quadrupole formalism is used to obtain a compact form of the solution that can be, with some exception, straightforwardly extended to multi-slab composite cylinders. The limiting cases of infinite thickness and infinite radius are also considered and solved.

Keywords: Periodic heat conduction; Thermal quadrupoles; Composite cylinder

1. Introduction

Unsteady heat conduction in homogeneous solids has been deeply analysed both for its inherent mathematical interest and for the vast consequences in many applied fields like laser heating, energy storage [1], building materials [2] etc. Many techniques used to evaluate thermophysical characteristics of solid materials, like microcalorimetry [3], or to measure convective heat transfer coefficient [4,5] rely on analytical solutions of transient heat conduction problems. The problem of transient multidimensional heat conduction is a challenging one and it has been faced by many different techniques, like finite integral transform, Green functions, orthogonal expansion, Laplace transform and also for the case of composite systems [6–10]. Periodic heat conduction is a relatively less investigated topic, despite of the significant implications for many applied fields, but many studies exist relative to periodic conduction in a 1-D homogeneous slab [11–14], generally limited to harmonic fluctuations and many of them related to the hyperbolic version of the heat equation, as the wave like behaviour can be better evidenced in a simple geometry. Studies of periodic conduction in applied fields also exist for some particular case, like for building walls and materials [15,16] or for techniques to measure thermophysical characteristics of the materials, like a.c. calorimetry [17] or 3-ω methods [18], and again for local convective heat transfer coefficient measurements [19]. Recently periodic conduction in non-homogeneous material was also considered for simple geometry [20]. The problem of periodic conduction in finite cylinder has recently been treated by Lu et al. [8–10] through the use of an original variable separation method, although only for a well defined kind of B.C. on the lateral surface and for harmonic forcing. The present work is intended to extend the method of Lu et al. [8] to treat the periodic, non-necessarily harmonic, conduction in a finite homogeneous cylinder for general periodic boundary conditions of first, second, and third kind with the restriction of uniformity on the lateral surface and radial symmetry on the bases. To the author's knowledge, no solution for this general case have been reported in the open literature. Moreover, the introduction of the thermal quadrupole formalism [21] allows a relatively straightforward extension of the results to the case of composite cylinders and, with some exception, the application of a relatively simple way to introduce the boundary condition directly on the solving formulae. Two significant cases are also treated, namely the semi-infinite cylinder of finite radius and the cylinder of finite thickness and infinite radius.

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Nomenclature

A_n, B_n	constants
Bi	Biot number: $\frac{hR}{k}$
F	forcing function
G	transfer function
h	heat transfer coefficient
H_n	Bessel series coefficients
J_0, J_1	Bessel functions
k	thermal conductivity
L	cylinder length
$\mathcal{M}_n$	quadrupole matrix
N_n	auxiliary functions
P	Fourier transform of thermal power fluctuations
Q	Fourier transform of heat flux fluctuations
q	heat flux
R	cylinder radius
r	radial coordinate
S	Fourier transform of temperature fluctuations
$\mathcal{T}$	period
T	temperature
t	time
u_n, v_n	Bessel series coefficients
X_n, Y_n, Z_n	complex functions
$\mathbf{Z}$	temperature-heat flux vector

Greek symbols

α	thermal diffusivity
β	non-dimensional complex frequency
γ_n	eigenvalues
$\delta(x)$	Dirac delta-function
δ_{jk}	Kronecker symbol
ε	truncation accuracy
η	non-dimensional radial coordinate
Θ	Hankel transform of S
λ_n	complex parameters
ξ	non-dimensional axial coordinate
σ	width of the Gaussian distribution
Φ	Fourier transform of forcing function
ω	angular frequency

2. Basic equations

Consider the case of a cylindrical homogeneous solid bar of radius R and length L, subject to periodic (non necessarily harmonic) thermal boundary conditions on its surfaces, with the restriction of uniformity on the lateral surface and radial symmetry on the bases. Introducing the non-dimensional coordinates: $\eta = \frac{r}{R}$; $\xi = \frac{x}{R}$, where r is the radial coordinate ($0 \leqslant r \leqslant R$) and x is the axial coordinate ($0 \leqslant x \leqslant L$), the Fourier equation can be written:

$$\frac{\partial T}{\partial t} = \frac{\alpha}{R^2}\left[\frac{\partial^2 T}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial T}{\partial \eta}\right)\right] \tag{1}$$

Boundary conditions of first, second and third kind can be expressed through linear combinations of temperature and heat fluxes on the boundary, nonlinearities are introduced when, for example, radiative boundary conditions are considered, however such conditions can be linearised whenever the surface temperature fluctuations are small compared to the time-averaged absolute temperature [22]. Then the general linear boundary conditions subject to the previously mentioned restrictions can be written as:

$$\begin{aligned}
&\xi = 0, \quad a_1 T(0,\eta,t) + a_2 q_\xi(0,\eta,t) = F_0(\eta,t)\\
&\xi = \xi_L, \quad b_1 T(\xi_L,\eta,t) + b_2 q_\xi(\xi_L,\eta,t) = F_L(\eta,t)\\
&\eta = 1, \quad c_1 T(\xi,1,t) + c_2 q_\eta(\xi,1,t) = F_R(t)\\
&\eta = 0, \quad \frac{\partial T}{\partial \eta}(\xi,0,t) = 0;
\end{aligned} \tag{2}$$

where $\mathbf{q} = (q_\xi, q_\eta) = -\frac{k}{R}(T_\xi, T_\eta)$, and the functions F_s (with $s = 0, L, R$) are periodic with time. After splitting the temperature and heat flux fields into time average and fluctuating parts and introducing the Fourier transforms of the latter as:

$$T(x,r,t) = T_a(x,r) + \int_{-\infty}^{+\infty} S(x,r,\omega)e^{i\omega t}\,d\omega$$

$$\mathbf{q}(x,r,t) = \mathbf{q}_a(x,r) + \int_{-\infty}^{+\infty} \mathbf{Q}(x,r,\omega)e^{i\omega t}\,d\omega$$

the Fourier equation (1) can be split into a time independent equation

$$\frac{\partial^2 T_a}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial T_a}{\partial \eta}\right) = 0 \tag{3}$$

and a transformed equation:

$$\beta S = \frac{\partial^2 S}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial S}{\partial \eta}\right) \tag{4}$$

with $\beta = \frac{i\omega R^2}{\alpha}$. The corresponding B.C. are easily found, but it will be seen later that the third of (2) needs a little discussion. Since the main difficulties in finding the analytic solution are related to the transformed equation (4), in the next session the discussion will be devoted to the solution of Eq. (4) while the solutions of the time independent equation (3) can be obtained referring to those of (4) by setting $\beta = 0$.

It should be pointed out that once the solution, in terms of the transformed field, is found, the solution in term of temperature field can be obtained by the inverse Fourier transformation. Although such transformation may not be always obtainable analytically, the use of ready available numeric inversion routines may yield the wanted solutions, with a high degree of accuracy, for any type of periodicity of the boundary condition. Moreover, the case of harmonic periodic boundary conditions is instead obtainable analytically observing that in such case

$$S(x,r,\omega) = \delta(\omega - \omega_0)S(x,r)$$

and

$$T(x,r,t)=\mathrm{Re}\{S(x,r,\omega)e^{i\omega t}\}$$

3. Solution of the transformed problem

Consider the splitting of the forcing functions F_s into time averages and fluctuating parts and introduce the Fourier transforms of the latter as:

$$F_s=F_{s,a}+\int_{-\infty}^{+\infty}\Phi_s e^{i\omega t}\,d\omega$$

then the B.C. for the transformed problem become:

$$\begin{aligned}
&\xi=0, \quad a_1S(0,\eta,\omega)+a_2Q_\xi(0,\eta,\omega)=\Phi_0(\eta,\omega)\\
&\xi=\xi_L, \quad b_1S(\xi_L,\eta,\omega)+b_2Q_\xi(\xi_L,\eta,\omega)=\Phi_L(\eta,\omega)\\
&\eta=1, \quad c_1S(\xi,1,\omega)+c_2Q_\eta(\xi,1,\omega)=\Phi_R(\omega)\\
&\eta=0, \quad \frac{\partial S}{\partial\eta}(\xi,0,\omega)=0;
\end{aligned} \tag{5}$$

It should be mentioned that the problem treated here is very similar to that treated by Lu et al. [8] and the method of finding the solution will follow a path similar to that suggested in [8]. It should be stressed anyway that the present work generalises that of Lu et al. [8] in that of extending it to more general boundary conditions, i.e. non-uniform boundary conditions on the cylinder bases and second and third kind on the lateral surface, while in [8] the solution was given for the single case of uniform first kind B.C. The extension to the case of a composite cylinder will then be obtained in a relatively simple way through the use of the thermal quadrupoles formalism. The third of (5) is quite important to define the form of the solution and it needs a little discussion as the nature of the solution of the transformed equation (4) depends on the choice of the values of c_1, c_2 and Φ_R. The B.C. for the non-homogeneous case ($\Phi_R\neq 0$) should be treated differently for the following cases:

$(c_1,c_2)=(1,0)$; $\Phi_R\neq 0$: first kind

$(c_1,c_2)=(0,1)$; $\Phi_R\neq 0$: second kind

$(c_1,c_2)=(1,-1/h)$; $\Phi_R\neq 0$: third kind

whereas the homogeneous B.C. ($\Phi_R=0$) can be seen as a particular simpler case (to notice that h has the meaning of a convective heat transfer coefficient).

Consider first the most general case of B.C. of third kind $c_1=1$, $c_2=-1/h$, the problem can be solved by the following procedure. Let first set a new unknown defined by:

$$\tilde S=S-\Phi_R$$

in order to homogenise the B.C.; then Eq. (4) becomes:

$$\frac{\partial^2\tilde S}{\partial\xi^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\tilde S}{\partial\eta}\right)=\beta\tilde S+\beta\Phi_R \tag{6}$$

Consider now the auxiliary problem:

$$\begin{aligned}
&\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial N}{\partial\eta}\right)=-\gamma^2N\\
&hN(1,\omega)+\frac{k}{R}\frac{\partial}{\partial\eta}N(1,\omega)=0; \qquad \frac{\partial}{\partial\eta}N(0,\omega)=0
\end{aligned} \tag{7}$$

which it is satisfied by the Bessel functions of first kind and zero order:

$$N_n(\eta)=J_0(\gamma_n\eta)$$

where the values of the parameter γ are the non-negative roots γ_n of the equation:

$$J_0(\gamma_n)-\frac{k}{hR}\gamma_nJ_1(\gamma_n)=0 \tag{8}$$

It is well known (see for example [23]) that the eigenvalues are all distinct and the eigensolutions N_n form a complete basis for the expansion of functions of η in Dini series. Then, consider the following possible form of the solution of (6):

$$\tilde S=\sum_{n=1}^{\infty}Y_n(\xi,\omega)J_0(\gamma_n\eta) \tag{9}$$

(to notice that $Y_n(\xi,\omega)$ are complex functions) that satisfies the third and fourth of (5) and substitute it into Eq. (6) obtaining, after noticing that:

$$1=\sum_{n=1}^{\infty}u_nJ_0(\gamma_n\eta); \quad u_n=\frac{2J_1(\gamma_n)}{\gamma_n(J_0^2(\gamma_n)+J_1^2(\gamma_n))}$$

a set of equation for the functions $Y_n(\xi,\omega)$:

$$\frac{\partial^2Y_n(\xi,\omega)}{\partial\xi^2}-(\gamma_n^2+\beta)Y_n(\xi,\omega)=\beta\Phi_Ru_n$$

whose solutions are:

$$Y_n=X_n-\frac{\beta\Phi_Ru_n}{(\gamma_n^2+\beta)}$$

with:

$$X_n(\xi,\omega)=A_ne^{\lambda_n\xi}+B_ne^{-\lambda_n\xi}; \quad \lambda_n=\sqrt{\gamma_n^2+\beta} \tag{10}$$

so that:

$$S=\sum_{n=1}^{\infty}X_n(\xi,\omega)J_0(\gamma_n\eta)+\Phi_R\left[1-\beta\sum_{n=1}^{\infty}\frac{u_n}{\gamma_n^2+\beta}J_0(\gamma_n\eta)\right] \tag{11}$$

The case of B.C. of first kind is simply a particular case of this problem, obtained after setting $c_2=0$, which defines the new set of eigenvalues through the equation

$$J_0(\gamma_n)=0$$

but the solution $S(\xi,\eta,\omega)$ has still the same general form (11).

Consider now the B.C. of second kind: $c_1=0$, $c_2=1$. The procedure is similar to that above reported but slightly more involved. To homogenise the B.C. on $\eta=1$ let set:

$$\tilde S=S+\Phi_R\frac{2R}{\beta k}\left(1+\frac{\beta\eta^2}{4}\right)$$

then Eq. (4) becomes:

$$\frac{\partial^2 \tilde{S}}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \tilde{S}}{\partial \eta}\right) = \beta \tilde{S} - \Phi_R \frac{2R}{k}\frac{\beta \eta^2}{4} \tag{12}$$

Consider again the set of function satisfying the problem posed by Eqs. (7) and (8), where now $c_1 = 0$ and $c_2 = 1$, choosing:

$$\tilde{S} = \sum_{n=1}^{\infty} Y_n(\xi, \omega) J_0(\gamma_n \eta)$$

and substituting into Eq. (12) after noticing that:

$$\eta^2 = \sum_{n=1}^{\infty} \upsilon_n J_0(\gamma_n \eta); \quad \upsilon_n = \begin{cases} \frac{1}{2} & \text{for } n = 1;\ \gamma_1 = 0 \\ \frac{4}{\gamma_n^2 J_0(\gamma_n)} & \text{for } n > 1;\ \gamma_1 > 0 \end{cases} \tag{13}$$

where the identity $\int_0^x J_0(\eta)\eta^3\,d\eta = [x^3 - 4x]J_1(x) + 2x^2 J_0(x)$ was used (see [23]), yields:

$$\frac{\partial^2 Y_n(\xi, \omega)}{\partial \xi^2} - (\gamma_n^2 + \beta) Y_n(\xi, \omega) = -\Phi_R \frac{R}{k}\frac{\beta}{2}\upsilon_n$$

whose solutions are

$$Y_n = X_n + \Phi_R \frac{R}{2k}\frac{\beta \upsilon_n}{\gamma_n^2 + \beta}$$

with X_n again given by (10). The solution has then the form

$$\begin{aligned} S &= \sum_{n=1}^{\infty} X_n J_0(\gamma_n \eta) \\ &\quad + \Phi_R \frac{R}{2k}\left\{\sum_{n=1}^{\infty} \frac{\beta \upsilon_n}{(\gamma_n^2 + \beta)} J_0(\gamma_n \eta) - \frac{4}{\beta}\left(1 + \frac{\beta \eta^2}{4}\right)\right\} \\ &= X_1(\xi) + \sum_{n=2}^{\infty} X_n(\xi) J_0(\gamma_n \eta) \\ &\quad + \Phi_R \frac{R}{2k}\left\{\sum_{n=2}^{\infty} \frac{\beta \upsilon_n}{(\gamma_n^2 + \beta)} J_0(\gamma_n \eta) - \frac{4}{\beta}\left(1 + \frac{\beta(\eta^2 - \frac{1}{2})}{4}\right)\right\} \end{aligned}$$

and the eigenvalues are the non-negative roots of the equation: $J_1(\gamma_n) = 0$.

Summarising the results, the general form of the solution is then:

$$S(\xi, \eta, \omega) = \sum_{n=1}^{\infty} X_n(\xi, \omega) J_0(\gamma_n \eta) + \Phi_R(\omega)\left[s(\eta) + \sum_{n=1}^{\infty} g_n(\omega) J_0(\gamma_n \eta)\right] \tag{14}$$

with $X_n(\xi, \omega)$ given by (10) and:

$$g_n(\omega) = \begin{cases} -\beta \frac{u_n}{\gamma_n^2 + \beta} & \text{1st kind B.C.} \\ \frac{R}{2k}\left\{\frac{\beta \upsilon_n}{\gamma_n^2 + \beta}\right\} & \text{2nd kind B.C.} \\ -\beta \frac{u_n}{\gamma_n^2 + \beta} & \text{3rd kind B.C.} \end{cases}$$

$$s(\eta) = \begin{cases} 1 & \text{1st kind B.C.} \\ -\frac{2R}{k\beta}\left(1 + \frac{\beta \eta^2}{4}\right) & \text{2nd kind B.C.} \\ 1 & \text{3rd kind B.C.} \end{cases} \tag{15}$$

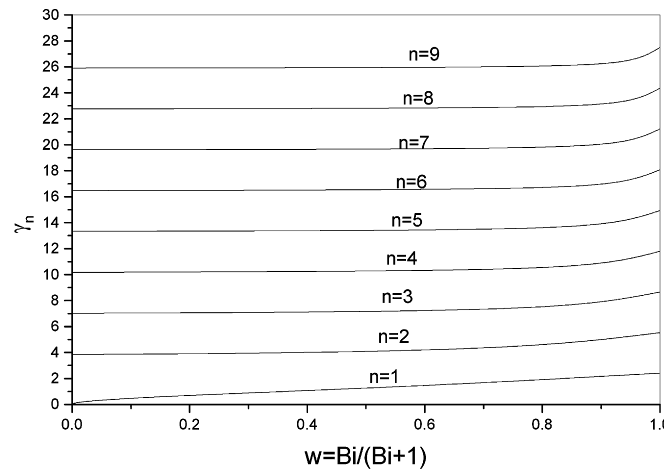

Fig. 1. Eigenvalues γ_n as a function of Bi.

To notice that the eigenvalues γ_n are the non-negative roots of Eq. (8), thus they are different for the three cases above reported. Introducing the Biot number: $Bi = \frac{hR}{k}$, the eigenvalues γ_n, see Eq. (8), are then function of Bi, and the particular cases of 1st and 2nd kind B.C. are those corresponding to the values $Bi = \infty$ and $Bi = 0$ respectively, Fig. 1 shows these functions for the first few values of n. To notice that the solution for the homogeneous B.C. case is found by setting $\Phi_R = 0$ into the general solution (14). The B.C. imposed by the first and second of (5) can now be easily satisfied by a proper choice of the constants A_n and B_n in (10) and the next session shows some examples of application of the above reported results for different B.C.

4. Example of application

As a first example consider the case of harmonic heating of the surface at $\xi = 0$, while the flux on surface $\xi = \xi_L$ is kept constant and the surface at $\eta = 1$ is under convective conditions with a constant fluid temperature. Let $\Phi_0(\eta)$ be the distribution of the amplitude fluctuation of the heat flux at $\xi = 0$ then the B.C. are:

$$Q_\xi(0, \eta) = \Phi_0(\eta); \quad S_\xi(\xi_L, \eta) = 0$$
$$S_\eta(\xi, 1) + Bi\, S(\xi, 1) = 0$$

where $Bi = \frac{hR}{k}$. From the general solution (14) this yields:

$$S(\xi, \eta) = \sum_{n=1}^{\infty} -\frac{H_n}{\lambda_n}\frac{\cosh[\lambda_n(\xi - \xi_L)]}{\sinh(\lambda_n \xi_L)} J_0(\gamma_n \eta) \tag{16}$$

where γ_n are the solution of the equation:

$$\gamma_n J_1(\gamma_n) = Bi\, J_0(\gamma_n)$$

and:

$$H_n = \frac{\int_0^1 \Phi_0(\eta) J_0(\gamma_n \eta)\eta\,d\eta}{\int_0^1 J_0^2(\gamma_n \eta)\eta\,d\eta}$$

Choosing a Gaussian distribution for the imposed heat flux fluctuation: $\Phi_0(\eta) = e^{-\eta^2/(2\sigma^2)}$, the case may represent that

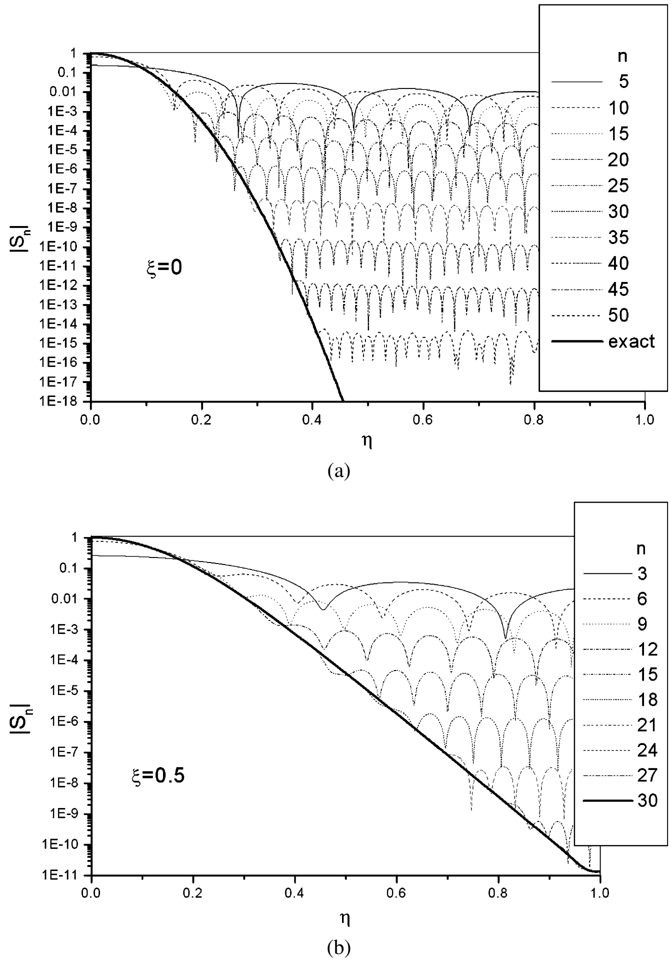

Fig. 2. (a) Absolute value of partial sums S_n at $\xi = 0$; (b) Absolute value of partial sums S_n at $\xi = \xi_L = 0.5$.

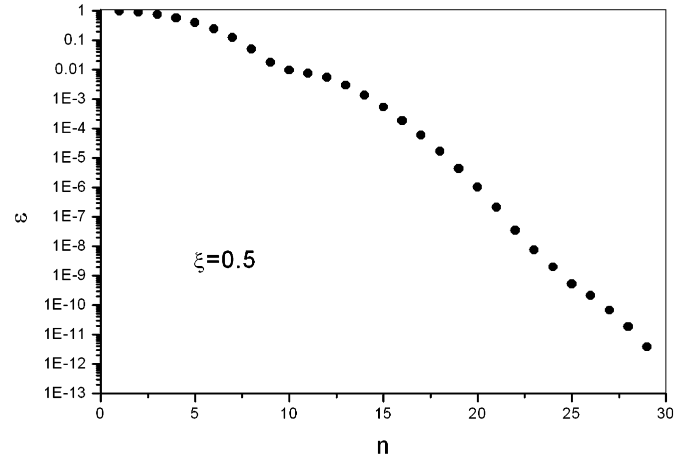

Fig. 3. Value of the summation error ε as defined in the text.

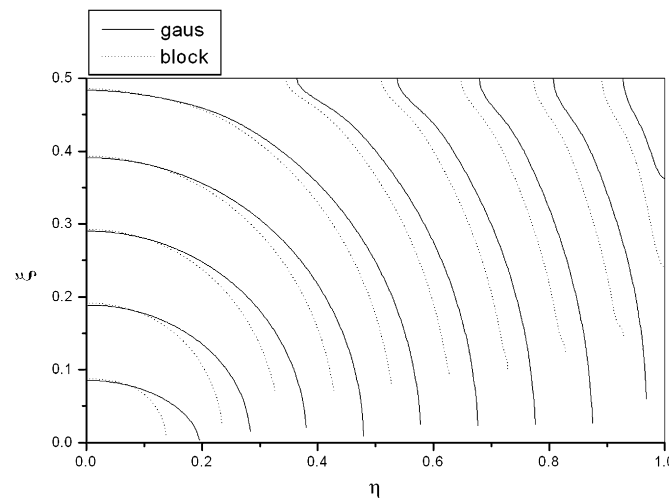

Fig. 4. "Wave" front for Gaussian and top-hat (block) distribution of the heat flux amplitude fluctuation imposed onto the surface at $\xi = 0$.

of laser heating of a cylindrical sample, to notice that in this case the coefficients H_n are approximated by (see [28]): $H_n = 2\sigma^2 e^{-(\gamma_n^2\sigma^2)/2}/(J_0^2(\gamma_n)[1 + (Bi/\gamma_n)^2])$, under the hypothesis $\sigma \ll 1$ (where σ is the "width" of the Gaussian distribution); actually, the accuracy of this approximation is better than 1% for $\sigma < 0.2$ and $n < 20$. The numerical example was set by choosing $\xi_L = 0.5$ and $Bi = 1$. The convergence of the series (16) in this case can be appreciated observing Fig. 2 that shows the absolute value of the partial sum $S_n(\xi, \eta) = \sum_{k=1}^{n} X_k(\xi) J_0(\gamma_k \eta)$ for $\xi = 0$ and $\xi = \xi_L = 0.5$, and from Fig. 3 where the parameter: $\varepsilon = \max(|S_n(0.5, \eta) - S_{30}(0.5, \eta)|)$ is shown as a function of n.

Fig. 4 reports also the distribution of the points where the phase difference with the input heat flux fluctuation is nil or a multiple of 2π, thus showing a sort of wave front location. The region where the curve are not reported is that where the field intensity is so small (order of 10^{-14}) that the zero-finding procedure used to detect the wave fronts does not give consistent results due to numerical inaccuracy. The figure reports also the results obtained using a top-hat distribution of the input heat flux intensity fluctuation on the surface at $\xi = 0$ (the width of the top-hat distribution was chosen so to produce the same total power fluctuation amplitude at $\xi = 0$, i.e. the integral $\int_0^1 \Phi_0(\eta)\eta\, d\eta$ is the same for both distributions).

As a second example consider the non-homogeneous problem set by imposing harmonic heat flux fluctuation on the lateral surface, with constant temperature on the surface at $\xi = 0$ and constant heat flux on $\xi = \xi_L$. The boundary conditions can then be written as:

$$S(0, \eta) = 0; \quad S_\xi(\xi_L, \eta) = 0; \quad S_\eta(\xi, 1) = -\frac{R}{k}\Phi_R$$

By applying these conditions to the general solution of the problem, remembering that γ_n are the solutions of $J_1(\gamma_n) = 0$, comprised $\gamma_1 = 0$, the following result is obtained:

$$S(\xi, \eta) = \sum_{n=1}^{\infty} \Phi_R \frac{R}{k} \frac{2}{\lambda_n^2 J_0(\gamma_n)} \frac{\cosh(\lambda_n(\xi - \xi_L))}{\cosh(\lambda_n \xi_L)} J_0(\gamma_n \eta) + \Phi_R \frac{R}{2k}\left\{\sum_{n=1}^{\infty} \frac{\beta v_n}{\gamma_n^2 + \beta} J_0(\gamma_n \eta) - \frac{4}{\beta}\left(1 + \frac{\beta\eta^2}{4}\right)\right\}$$

(see also Eq. (13)). It is interesting to observe that the x-component of heat flux Fourier transform is:

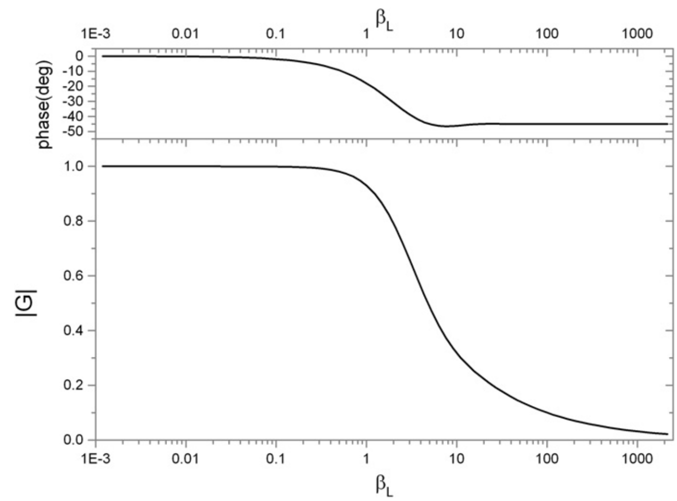

Fig. 5. Bode diagram for the ratio $G = (P_{\xi=0})/P_R$.

$$Q_\xi = -\frac{k}{R} S_\xi(\xi,\eta)$$
$$= -\sum_{n=1}^{\infty} \frac{2}{\lambda_n J_0(\gamma_n)} \Phi_R \frac{\sinh(\lambda_n(\xi-\xi_L))}{\cosh(\lambda_n \xi_L)} J_0(\gamma_n \eta)$$

then, the transform of the total power on the isothermal basis ($\xi = 0$) is simply:

$$P_{\xi=0} = 2\pi R^2 \int_0^1 Q_\xi(0,\eta)\eta\, d\eta$$
$$= 2\pi R^2 \sum_{n=1}^{\infty} \frac{2\tanh(\lambda_n \xi_L)}{\lambda_n J_0(\gamma_n)} \Phi_R \int_0^1 J_0(\gamma_n \eta)\eta\, d\eta$$
$$= 2\pi R^2 \frac{\tanh(\sqrt{\beta}\,\xi_L)}{\sqrt{\beta}} \Phi_R \tag{17}$$

Since the Fourier transform of the total power on the lateral basis is: $P_R = \Phi_R 2\pi RL$, the transfer function $G(\omega) = \frac{P_{\xi=0}}{P_R}$ is simply:

$$G(\omega) = \frac{\tanh(\sqrt{\beta}\,\xi_L)}{\sqrt{\beta}\,\xi_L}$$
$$\sqrt{\beta}\,\xi_L = \sqrt{\frac{i\omega L^2}{\alpha}}$$

and Fig. 5 shows the Bode plots. The fluctuations are in phase for low fluctuation frequencies and in quadrature for large fluctuation frequency and the maximum phase difference (46.6 deg) is achieved for $\sqrt{\beta}\xi_L \simeq 7.9$. The solutions in terms of Fourier transform can then be used to obtain solution in time domain for virtually any periodic non-harmonic input. As an example, consider the previous case where the input heat flux on the lateral surface is intermittent with period $\mathcal{T}$, i.e.:

$$q_R(t) = q_0 w(t) \quad \text{with } w(t) = \begin{cases} 1, & |t| < \frac{a}{2} \\ 0, & \frac{a}{2} < |t| \leqslant |\frac{\mathcal{T}}{2}| \end{cases}$$

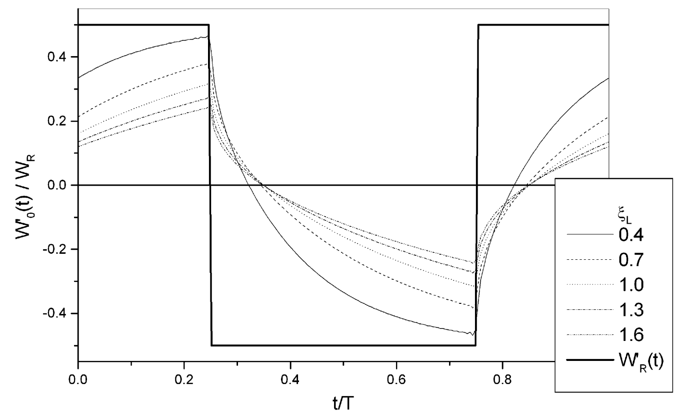

Fig. 6. Normalised heat power fluctuation on the surface $\xi = 0$ for different values of the non-dimensional cylinder length and for $a = \mathcal{T}/2$.

then the heat flux fluctuation is: $q'_R(t) = q_R(t) - q_0 \frac{a}{\mathcal{T}}$ (see Fig. 6 showing the case $a = \mathcal{T}/2$). In this case, the Fourier transform of the heat flux fluctuation on the lateral surface is:

$$\Phi_R = q_0 \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0)$$

with (see [27]): $\omega_0 = \frac{2\pi}{\mathcal{T}}$, $c_0 = 0$, $c_m = \frac{2}{\mathcal{T}} \frac{\sin(m\omega_0 a/2)}{m\omega_0}$ for $m \neq 0$. Then the corresponding Fourier transform of the heat power fluctuation on the surface at $\xi = 0$ is given by Eq. (17) and the solution in the time domain is:

$$W_0(t) = \int_{-\infty}^{+\infty} P_{\xi=0}(\omega) e^{i\omega t}\, d\omega$$
$$= \int_{-\infty}^{+\infty} 2\pi R^2 \xi_L G(\omega) q_0 \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0) e^{i\omega t}\, d\omega$$
$$= W_R \sum_{n=-\infty}^{+\infty} c_n G(n\omega_0) e^{in\omega_0 t}$$

where $W_R = 2\pi RLq_0$ is the amplitude fluctuation of the heat power on the lateral surface, and Fig. 6 shows the heat power fluctuation on the surface at $\xi = 0$, for different values of ξ_L, for the case $a = \mathcal{T}/2$ and $\mathcal{T} = \frac{R^2}{\alpha}$. As expected, the increase of the cylinder length, increasing the thermal inertia, decreases the amplitude fluctuation of the heat power on the surface $\xi = 0$, consistently with the results shown in the Bode plots of Fig. 5.

It is possible to develop a general way to introduce the various kind of B.C. on the surface at $\xi = 0$ and $\xi = \xi_L$ into the solving formulae, and this can be done in a relatively simple way resorting on the thermal quadrupole formalism, as it will be shown in the next section.

5. The solution under the thermal quadrupole formalism

The thermal quadrupole formalism (see [21]) is a useful and compact way to represent the solution of a conduction problem and it allows a relatively simple treatment of conduction in composite systems. After defining $Z_n(\xi,\omega) = -\frac{k}{R} X_{n,\xi}(\xi,\omega)$, the functions (10) can be written under the form:

$$X_n(\xi,\omega) = X_n(0,\omega)\cosh(\lambda_n\xi) - Z_n(0,\omega)\frac{R}{k\lambda_n}\sinh(\lambda_n\xi)$$
$$Z_n(\xi,\omega) = -\frac{k}{R}\lambda_n X_n(0,\omega)\sinh(\lambda_n\xi) + Z_n(0,\omega)\cosh(\lambda_n\xi) \tag{18}$$

and setting:

$$\mathbf{Y}_n(\xi,\omega) = \begin{bmatrix} X_n \\ Z_n \end{bmatrix}$$
$$\mathcal{M}_n(\xi,\omega) = \begin{bmatrix} \cosh(\lambda_n\xi) & -\frac{R}{k\lambda_n}\sinh(\lambda_n\xi) \\ -\frac{k\lambda_n}{R}\sinh(\lambda_n\xi) & \cosh(\lambda_n\xi) \end{bmatrix}$$

Eqs. (18) can be written in the compact matrix form:

$$\mathbf{Y}_n(\xi,\omega) = \mathcal{M}_n(\xi,\omega)\mathbf{Y}_n(0,\omega) \tag{19}$$

that is the basis of the thermal quadrupole approach. The linearity of the problem allows also to introduce the B.C. into the general solution in a simple way (see also [24]). After defining:

$$\hat{\Phi}_0(\eta,\omega) = \Phi_0(\eta,\omega) - \Phi_R a_1 s(\eta) = \sum_{n=1}^{\infty} f_{0,n}(\omega) J_0(\gamma_n\eta)$$
$$\hat{\Phi}_L(\eta,\omega) = \Phi_L(\eta,\omega) - \Phi_R b_1 s(\eta) = \sum_{n=1}^{\infty} f_{L,n}(\omega) J_0(\gamma_n\eta)$$

where the series expansion is allowed by the completeness of the set $J_0(\gamma_n\eta)$, the B.C. on $\xi = 0$ and $\xi = \xi_L$ can be written under the general form:

$$\begin{aligned} &a_1 X_n(0,\omega) + a_2 Z_n(0,\omega) \\ &\quad = f_{0,n}(\omega) - a_1\Phi_R(\omega)g_n(\omega) = p_{0,n}(\omega) \\ &b_1 X_n(\xi_L,\omega) + b_2 Z_n(\xi_L,\omega) \\ &\quad = f_{L,n}(\omega) - b_1\Phi_R(\omega)g_n(\omega) = p_{L,n}(\omega) \end{aligned} \tag{20}$$

Setting

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}; \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Eqs. (20) can be written in compact form:

$$\mathbf{a}^T\mathbf{Y}_n(0) = p_{0,n}; \qquad \mathbf{b}^T\mathbf{Y}_n(\xi_L) = p_{L,n} \tag{21}$$

and using (19), the second of (21) becomes:

$$\mathbf{b}^T\mathcal{M}_n(\xi_L,\omega)\cdot\mathbf{Y}_n(0) = p_{L,n}$$

Defining:

$$\mathbf{p}_n = \begin{bmatrix} p_{0,n} \\ p_{L,n} \end{bmatrix}; \quad \mathbf{d}^T = \mathbf{b}^T\mathcal{M}_n(\xi_L); \quad \mathcal{C}_n = \begin{bmatrix} a_1 & a_2 \\ d_1 & d_2 \end{bmatrix} \tag{22}$$

the B.C. (20) can be written (under the condition $\det(\mathcal{C}_n) \neq 0$) as:

$$\mathbf{Y}_n(0) = \mathcal{C}_n^{-1}\cdot\mathbf{p}_n \tag{23}$$

and

$$\mathbf{Y}_n(\xi) = \mathcal{M}_n(\xi)\cdot\mathcal{C}_n^{-1}\cdot\mathbf{p}_n$$

which yields, in a compact form, the functions $\mathbf{Y}_n(\xi)$ in terms of the forcing functions.

The solution of the transformed problem (4) can then be written in compact form after defining:

$$\mathbf{Z}(\xi,\eta) = \begin{bmatrix} S(\xi,\eta) \\ Q(\xi,\eta) \end{bmatrix}$$
$$\mathbf{g}_n(\omega) = \begin{bmatrix} g_n(\omega) \\ 0 \end{bmatrix}; \qquad \mathbf{s}(\eta) = \begin{bmatrix} s(\eta) \\ 0 \end{bmatrix}$$
$$\mathbf{H}_n(\xi) = \mathbf{Y}_n(\xi) + \Phi_R\mathbf{g}_n = \mathcal{M}_n(\xi)\cdot\mathcal{C}_n^{-1}\cdot\mathbf{p}_n + \Phi_R\mathbf{g}_n \tag{24}$$

then:

$$\mathbf{Z}(\xi,\eta) = \sum_{n=1}^{\infty}\mathbf{H}_n(\xi)J_0(\gamma_n\eta) + \Phi_R\,\mathbf{s}(\eta)$$

6. Extension to composite cylinders

The thermal quadrupole formalism allows an almost straightforward extension of the results to m-slab composite cylinder. The solution in each slab can be written as:

$$\begin{aligned} \mathbf{Z}^{(m)}(\xi,\eta) &= \begin{bmatrix} S^{(m)}(\xi,\eta) \\ Q^{(m)}(\xi,\eta) \end{bmatrix} \\ &= \sum_{n=1}^{\infty}\big[\mathbf{Y}_n^{(m)}(\xi) + \Phi_R\mathbf{g}_n^{(m)}\big]J_0\big(\gamma_n^{(m)}\eta\big) + \Phi_R\,\mathbf{s}^{(m)}(\eta) \end{aligned} \tag{25}$$

where the index m is relative to the m-th slab, the coordinate ξ must now be considered as "local", i.e. for the m-th slab it always spans between 0 and $\xi_m = \frac{L_m}{R}$, $\gamma_n^{(m)} = \gamma_n(Bi_m)$ with $Bi_m = \frac{hR}{k_m}$ (k_m being the conductivity of the m-th slab). The vectors $\mathbf{Y}_n^{(m)}(\xi)$ can be written as:

$$\mathbf{Y}_n^{(m)}(\xi) = \begin{bmatrix} X_n^{(m)}(\xi,\eta) \\ W_n^{(m)}(\xi,\eta) \end{bmatrix} = \mathcal{M}_n^{(m)}(\xi)\cdot\mathbf{Y}_n^{(m)}(0) \tag{26}$$

and the interface conditions can generally be written as:

$$\mathbf{Z}_n^{(m)}(0,\eta) = \mathbf{Z}_n^{(m-1)}(\xi_{m-1},\eta) \tag{27}$$

6.1. The homogeneous case $\Phi_R = 0$

Also in this case, the B.C. at $\eta = 1$ set the nature of the problem and the three different kinds must be treated separately. In fact, for the 1st and 2nd kind, the eigenvalues $\gamma_n^{(k)}$ are the same for all the slabs (in fact $Bi = \infty$ and $Bi = 0$ respectively, for all the slabs) while they are different for the 3rd kind B.C. For the 1st and 2nd kind, the interface conditions simplify to:

$$\mathbf{Y}_n^{(m)}(0) = \mathbf{Y}_n^{(m-1)}(\xi_{m-1}) \tag{28}$$

and applying repeatedly Eq. (26) one obtains:

$$\mathbf{Y}_n^{(m)}(\xi) = \mathcal{H}_n^{(m)}(\xi)\mathbf{Y}_n^{(1)}(0) \tag{29}$$

where:

$$\mathcal{H}_n^{(m)}(\xi) = \mathcal{M}_n^{(m)}(\xi)\cdot\mathcal{M}_n^{(m-1)}(\xi_{m-1})\cdots\mathcal{M}_n^{(1)}(\xi_1)$$

The 3rd kind B.C. on $\eta = 1$ imposes different sets of eigenvalues and eigenfunctions for each slab, and Eq. (27) becomes:

$$\sum_{n=1}^{\infty} \mathbf{Y}_n^{(m)}(0) J_0\big(\gamma_n^{(m)}\eta\big) = \sum_{n=1}^{\infty} \mathbf{Y}_n^{(m-1)}(\xi_{m-1}) J_0\big(\gamma_n^{(m-1)}\eta\big) \quad (30)$$

and does not simplifies to (28). However the completeness of the eigenfunction set allows to write:

$$J_0\big(\gamma_n^{(m-1)}\eta\big) = \sum_{l=1}^{\infty} a_{n,l}^{m-1,m} J_0\big(\gamma_l^{(m)}\eta\big) \quad (31)$$

where (see [28]):

$$a_{p,l}^{m-1,m} = \frac{\int_0^1 J_0(\gamma_p^{(m-1)}\eta) J_0(\gamma_l^{(m)}\eta)\eta\, d\eta}{\int_0^1 [J_0(\gamma_l^{(m)}\eta)]^2 \eta\, d\eta} = 2\frac{(Bi^{(m)} - Bi^{(m-1)}) J_0(\gamma_p^{(m-1)})}{[(\gamma_l^{(m)})^2 - (\gamma_p^{(m-1)})^2][1 + (Bi^{(m)}/\gamma_l^{(m)})^2] J_0(\gamma_l^{(m)})} \quad (32)$$

Then, Eq. (30) becomes:

$$\sum_{n=1}^{\infty} \mathbf{Y}_n^{(m)}(0) J_0\big(\gamma_n^{(m)}\eta\big) = \sum_{n=1}^{\infty} \left\{ \sum_{p=1}^{\infty} a_{p,n}^{m-1,m} \mathbf{Y}_p^{(m-1)}(\xi_{m-1}) \right\} J_0\big(\gamma_n^{(m)}\eta\big)$$

yielding:

$$\mathbf{Y}_n^{(m)}(0) = \sum_{p=1}^{\infty} a_{p,n}^{m-1,m} \mathbf{Y}_p^{(m-1)}(\xi_{m-1})$$

and resorting on Eq. (26):

$$\mathbf{Y}_n^{(m)}(0) = \sum_{p=1}^{\infty} \mathcal{B}_{p,n}^{m-1,m}(\xi_{m-1}) \cdot \mathbf{Y}_p^{(m-1)}(0) \quad (33)$$

with:

$$\mathcal{B}_{p,n}^{m-1,m}(\xi) = a_{p,n}^{m-1,m} \mathcal{M}_p^{(m-1)}(\xi)$$

By applying repeatedly Eq. (33) and using (26) the following result is obtained:

$$\mathbf{Y}_{n_m}^{(m)}(\xi) = \sum_{n_1=1}^{\infty} \hat{\mathcal{H}}_{n_m,n_1}^{(m)}(\xi) \cdot \mathbf{Y}_{n_1}^{(1)}(0) \quad (34)$$

with

$$\hat{\mathcal{H}}_{n_m,n_1}^{(m)}(\xi) = \mathcal{M}_{n_m}^{(m)}(\xi) \cdot \sum_{n_2=1}^{\infty} \cdots \sum_{n_{m-1}=1}^{\infty} \mathcal{B}_{n_{m-1},n_m}^{m-1,m}(\xi_{m-1}) \cdots \mathcal{B}_{n_2,n_3}^{2,3}(\xi_2) \cdot \mathcal{B}_{n_1,n_2}^{1,2}(\xi_1)$$

Eq. (34) is the equivalent of (29) for the case of B.C. of third kind. It should be noticed that in case the eigenvalues are the same for all slabs the coefficients $\alpha_{p,n}^{m,m-1}$ become (see Eq. (32)): $\alpha_{p,n}^{m,m-1} = \delta_{p,n}$, then $\hat{\mathcal{H}}_{n_m,n_1}^{(m)}(\xi) = \delta_{n_m,n_1}\mathcal{H}_n^{(m)}(\xi)$ recovering the same result that holds when B.C. of 1st or 2nd kind are imposed on the surface $\eta = 1$.

6.2. The non-homogeneous case: $\Phi_R \neq 0$

The non-homogeneous case is slightly more involved. Generally, the interface conditions are still given by Eq. (27) but the conditions on $\eta = 1$ may produce inconsistency. Precisely, among the non-homogeneous B.C. at $\eta = 1$ only the 1st kind B.C. is unconditionally admissible and this is the only one that will be treated here. The interface conditions can then be written:

$$\mathbf{Y}_n^{(m)}(0) = \mathbf{Y}_n^{(m-1)}(\xi_{m-1}) + \Phi_R\big(\mathbf{g}_n^{(m-1)} - \mathbf{g}_n^{(m)}\big) \quad (35)$$

and using Eq. (26):

$$\mathbf{Y}_n^{(m)}(\xi) = \mathcal{M}_n^{(m)}(\xi)\mathbf{Y}_n^{(m-1)}(\xi_{m-1}) + \Phi_R \mathcal{M}_n^{(m)}(\xi)\big(\mathbf{g}_n^{(m-1)} - \mathbf{g}_n^{(m)}\big) \quad (36)$$

By applying repeatedly Eq. (36) the following rule is easily found:

$$\mathbf{Y}_n^{(m)}(\xi) = \mathcal{H}_n^{(m)}(\xi)\mathbf{Y}_n^{(1)}(0) + \Phi_R \mathcal{M}_n^{(m)}(\xi) \sum_{l=2}^{k} \mathcal{N}_n^{(m-1,l)}\big(\mathbf{g}_n^{(l-1)} - \mathbf{g}_n^{(l)}\big) \quad (37)$$

with

$$\mathcal{N}_n^{(m-1,l)} = \mathcal{M}_n^{(m-1)}(\xi_{m-1}) \cdot \mathcal{M}_n^{(m-2)}(\xi_{k-2}) \cdots \mathcal{M}_n^{(l)}(\xi_l) = \mathcal{H}_n^{(m-1)}(\xi_{m-1}) \cdot \big\{\mathcal{H}_n^{(l-1)}(\xi_{l-1})\big\}^{-1}$$

to notice that $\det \mathcal{M}_n^{(m-1)} = 1$ and $\{\mathcal{H}_n^{(m)}(\xi_m)\}^{-1}$ always exists. Eq. (37) generalises Eq. (29) to the non-homogeneous ($\Phi_R \neq 0$) case.

The B.C. can again be introduced into the general solution following a procedure similar to that seen in the previous section, observing that:

$$\mathbf{a}^T \mathbf{Y}_n^{(1)}(0) = p_{0,n}; \qquad \mathbf{b}^T \mathbf{Y}_n^{(m)}(\xi_m) = p_{L,n} \quad (38)$$

and referring to Eq. (20) for the definition of the functions $p_{s,n}$. From Eq. (37):

$$\mathbf{Y}_n^{(m)}(\xi_m) = \mathcal{H}_n^{(m)}(\xi_m)\mathbf{Y}_n^{(1)}(0) + \Phi_R \sum_{l=2}^{m} \mathcal{N}_n^{(m,l)}\big(\mathbf{g}_n^{(l-1)} - \mathbf{g}_n^{(l)}\big)$$

the second of (38) becomes:

$$\mathbf{b}^T \mathcal{H}_n^{(m)}(\xi_m)\mathbf{Y}_n^{(1)}(0) = p_{L,n} - \Phi_R b_1 \sum_{l=2}^{m} \mathcal{N}_n^{(m,l)}\big(g_n^{(l-1)} - g_n^{(l)}\big)$$

Defining

$$\mathbf{q}_n = \begin{bmatrix} p_{0,n} \\ p_{L,n} - \Phi_R b_1 \sum_{l=2}^{m} \mathcal{N}_n^{(m,l)}\big(g_n^{(l-1)} - g_n^{(l)}\big) \end{bmatrix}$$

we can now write (see the analogous equation (23))

$$\mathbf{Y}_n^{(1)}(0) = \mathcal{C}_n^{-1}\mathbf{q}_n \quad (39)$$

thus obtaining:

$$\mathbf{Y}_n^{(m)}(\xi) = \mathcal{H}_n^{(m)}(\xi)\left\{\mathcal{C}_n^{-1}\mathbf{q}_n + \Phi_R \sum_{l=2}^{m}\left(\mathcal{H}_n^{(l-1)}(\xi_{l-1})\right)\left(\mathbf{g}_n^{(l-1)} - \mathbf{g}_n^{(l)}\right)\right\} \tag{40}$$

Finally the field in each slab can be calculated using Eq. (25). This particular form of the general solution has the noticeable advantage of a relatively simple implementation when numerical results are needed. To notice that for the homogeneous cases with B.C. of 1st and 2nd kind equation (40) holds by setting $\Phi_R = 0$ everywhere. For the homogeneous case with B.C. of 3rd kind the procedure is not anymore applicable as in this case Eqs. (38) become:

$$\mathbf{a}^T \sum_{n_1=1}^{\infty} \delta_{n,n_1}\mathbf{Y}_{n_1}^{(1)}(0) = p_{0,n}$$

$$\mathbf{b}^T \sum_{n_1=1}^{\infty} \mathcal{H}_{n,n_1}^{(m)}(\xi_m)\mathbf{Y}_{n_1}^{(1)}(0) = p_{L,n}$$

and there is not any equivalent of Eq. (39). However, Eq. (34) allows to relate the solution in each slab to the B.C. on $\xi = 0$ or $\xi = \xi_L$ in compact form, which is the basis for the application of the normal thermal quadrupole formalism to this problem.

7. Two special cases

There are two interesting limiting cases to be analysed apart, as they need a different approach to obtain the solution and they complete the problem, namely: the semi-infinite cylinder of finite radius and the finite cylinder of infinite radius. These two cases are solved in the next subsections.

7.1. The semi-infinite cylinder of finite radius

Consider a semi-infinite cylindrical ($0 \leqslant x \leqslant \infty$) bar of finite radius R. The transformed problem (4) can be solved using the same procedure above reported, again the general solution can be written as:

$$S(\xi,\eta,\omega) = \sum_{n=1}^{\infty} X_n(\xi,\omega)J_0(\gamma_n\eta) + \Phi_R\left[\sum_{n=1}^{\infty} g_n(\omega)J_0(\gamma_n\eta) + s(\eta)\right]$$

where g_n and γ_n depend on the kind of B.C. on the lateral surface (see Eq. (15)). The solutions X_n have the general form (10) and the finiteness of their value at infinite implies that $A_n = 0$, and this also implies that they vanish at infinite. This means that the solution at infinite must be:

$$S(\infty,\eta,\omega) = \Phi_R\left[\sum_{n=1}^{\infty} g_n(\xi,\omega)J_0(\gamma_n\eta) + s(\eta)\right]$$

which is the solution that holds for an infinite bar subject to periodic uniform boundary conditions. The B.C. at $\xi = 0$, after setting

$$\hat{\Phi}_0 = \Phi_0 - \Phi_R a_1 s(\eta) = \sum_{n=1}^{\infty} f_{0,n}(\omega)J_0(\gamma_n\eta)$$

become:

$$a_1 X_n(0,\omega) + a_2 Z_n(0,\omega) = f_{0,n}(\omega) - a_1\Phi_R(\omega)g_n(\omega) = p_{0,n}(\omega)$$

and they are satisfied setting:

$$X_n(0,\omega) = \frac{p_{0,n}(\omega)}{a_1 + a_2 k\lambda_n/R}$$

It is worth to notice that the x-component of the heat flux transform can be written:

$$Q(\xi,\eta,\omega) = \sum_{n=1}^{\infty} \frac{k\lambda_n}{R} X_n(\xi,\omega)J_0(\gamma_n\eta)$$

and for any reasonable distribution of S on the surface at $\xi = 0$, the coefficients $f_{0,n}$ tend to zero for sufficiently large values of n, then there exists a value of β large enough that for all the $f_{0,n}$ that are significantly different from zero the approximation: $\lambda_n \simeq \sqrt{\beta}$ holds. In this case, for $\Phi_R = 0$ (homogeneous B.C. on the lateral surface):

$$Q(\xi,\eta,\omega) \simeq \sqrt{\beta}\sum_{n=1}^{\infty} X_n(\xi,\omega)J_0(\gamma_n\eta) = \sqrt{\beta}\,S(\xi,\eta,\omega)$$

which is exactly the relation that holds for the 1-D semi-infinite solid (see for example [24]), that means that the semi-infinite solid approximation holds also locally (i.e. for every η) for β sufficiently large.

7.2. The finite cylinder of infinite radius

This is the case of a plane slab of finite thickness ($0 \leqslant x \leqslant L$) and infinite on the other dimensions, but characterised by radial symmetric boundary conditions on its surfaces. This problem needs an alternative approach as now it is not anymore possible to set an eigenvalue problem. The B.C. for $R = \infty$ is now simply

$$S(\xi,\infty,\omega) = 0$$

while the first two of (5) still apply. Let consider the Hankel transform (some time called Fourier–Bessel, see for example [25]) of the field S, defined as:

$$\Theta(\xi,\gamma,\omega) = \int_0^{\infty} S(\xi,\eta,\omega)J_0(\gamma\eta)\eta\,d\eta$$

where now $\eta = \frac{r}{L}$, $\xi = \frac{x}{L}$, it is easily shown [26] that:

$$S(\xi,\eta,\omega) = \int_0^{\infty} \Theta(\xi,\gamma,\omega)J_0(\gamma\eta)\gamma\,d\gamma \tag{41}$$

satisfies the B.C. at $\eta = 0$ and $\eta = \infty$. Substituting (41) into Eq. (4) (where now $\beta = \frac{i\omega L^2}{\alpha}$) one obtains:

$$\frac{\partial^2\Theta(\xi,\gamma,\omega)}{\partial\xi^2}-(\gamma^2+\beta)\Theta(\xi,\gamma,\omega)=0$$

whose general solution is:

$$\Theta=Ae^{\lambda\xi}+Be^{-\lambda\xi}$$

with $\lambda=\sqrt{\gamma^2+\beta}$. From here on, the procedure based on the thermal quadrupole formalism can be applied, writing the B.C. on $\xi=0$ and $\xi=1$ under the general form:

$$\begin{aligned}a_1\Theta(0,\gamma,\omega)+a_2\Gamma(0,\gamma,\omega)&=f_0(\gamma,\omega)\\ b_1\Theta(1,\gamma,\omega)+b_2\Gamma(1,\gamma,\omega)&=f_L(\gamma,\omega)\end{aligned}\tag{42}$$

where $\Gamma=-\frac{k}{L}\Theta_\xi$, and

$$f_s(\gamma,\omega)=\int_0^\infty\Phi_s(\eta,\omega)J_0(\gamma\eta)\eta\,d\eta$$

Defining

$$\boldsymbol{\Psi}=\begin{bmatrix}\Theta\\ \Gamma\end{bmatrix};\quad \mathcal{M}(\xi,\gamma,\omega)=\begin{bmatrix}\cosh(\lambda\xi) & -\frac{R}{k\lambda}\sinh(\lambda\xi)\\ -\frac{k\lambda}{R}\sinh(\lambda\xi) & \cosh(\lambda\xi)\end{bmatrix}$$

and following the previously described procedure, the following solution is found:

$$\Psi(\xi,\gamma,\omega)=\mathcal{M}(\xi,\gamma,\omega)\,\mathcal{C}^{-1}\cdot\mathbf{p}$$

with

$$\mathbf{p}=\begin{bmatrix}f_0\\ f_L\end{bmatrix};\quad \mathbf{d}^T=\mathbf{b}^T\mathcal{M}(1,\gamma,\omega);\quad \mathcal{C}=\begin{bmatrix}a_1 & a_2\\ d_1 & d_2\end{bmatrix}$$

The solution of the problem in the physical space can then be written as:

$$\begin{aligned}\mathbf{Z}(\xi,\eta,\omega)&=\int_0^\infty\boldsymbol{\Psi}(\xi,\gamma,\omega)J_0(\gamma\eta)\gamma\,d\gamma\\ &=\int_0^\infty\mathcal{M}(\xi,\gamma,\omega)\mathcal{C}^{-1}\cdot\mathbf{p}J_0(\gamma\eta)\gamma\,d\gamma\end{aligned}$$

The case of semi-infinite solid with non-uniform (radial symmetric) B.C. at $x=0$ can be seen as a particular case of this problem obtained when L goes to infinite. In such case anyway the transformed equation (4) may be written:

$$\frac{\partial^2 S}{\partial\xi^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial S}{\partial\eta}\right)=iS$$

where now the coordinates x and r are non-dimensionalised by the penetration depth [21]: $d_p=\sqrt{\frac{\alpha}{\omega}}$, i.e. $\xi=x\sqrt{\frac{\omega}{\alpha}}$, $\eta=r\sqrt{\frac{\omega}{\alpha}}$. The condition at $\xi=\infty$ and the general B.C. at $\xi=0$:

$$a_1S(0,\eta,\omega)-a_2k\sqrt{\frac{\omega}{\alpha}}S_\xi(0,\eta,\omega)=\Phi_0(\eta,\omega)$$

yield the general solution:

$$S(\xi,\eta,\omega)=\int_0^\infty\frac{f_0(\gamma,\omega)}{a_1+a_2k\lambda/d_p(\omega)}e^{-\lambda\xi}J_0(\gamma\eta)\gamma\,d\gamma$$

with

$$f_s(\gamma,\omega)=\int_0^\infty\Phi_0(\eta,\omega)J_0(\gamma\eta)\eta\,d\eta$$

8. Conclusions

The problem of periodic conduction in a finite cylindrical slab was studied. Analytical solutions of the steady periodic problem were found for quite general kinds of boundary conditions with the sole limitation of uniform conditions (of any kind) on the lateral surface and cylindrical symmetry on the bases. Extension of the solution to the case of cylindrical composite slab was proposed on the framework of thermal quadrupole formalism and general solutions for any kind of complexity were then found in a form that may easily be implemented for numerical evaluation. The two limiting cases of semi-infinite cylinder of finite radius and of finite cylinder of infinite radius were also treated and solved, the case of periodic conduction in a semi-infinite solid with non-uniform cylindric symmetry boundary conditions can be seen as a particular case of the last problem and an explicit solution was given.

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